Pergamon

Int. J. Heat Mass Transfer. Vol. 39, No. 6, pp. 1287–1295, 1996
Published by Elsevier Science Ltd
Printed in Great Britain
0017-9310/96 $15.00+0.00

0017-9310(95)00146-8

The turbulent Prandtl number in the near-wall region for low-Prandtl-number gas mixtures

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(*Received* 5 *April* 1994 *and in final form* 5 *April* 1995)

Abstract—Results of experiments for binary gas mixtures, with molecular Prandtl numbers in the range 0.18–0.7 in flow in a smooth circular tube with constant transport properties were employed to examine models hypothesized for the distribution of the turbulent Prandtl number, $Pr_t\{y^+, Pr, Re\}$. These models included approximations of predictions from recent direct numerical simulations of turbulent duct flows. Data for a range of molecular Prandtl numbers for 'high' Reynolds numbers, $\sim 3 \times 10^4 < Re < \sim 1 \times 10^5$, were obtained earlier by the authors by utilizing mixtures formed by helium or hydrogen with xenon, argon or carbon dioxide. For the range $0.21 < Pr < 0.72$, Reynolds analogy and the variable $Pr_t\{y^+, Pr, Re\}$ model by Kays agreed with downstream measurements to within about 5%.

1. BACKGROUND

The concept of a turbulent Prandtl number is useful to predict heat transfer to or from the wall in forced turbulent flows via relatively simple numerical techniques; a number of reviews and texts have been devoted to its application [1–5]. For gases, the dominant thermal resistance is typically concentrated in the viscous layer, defined as $y^+ < \sim 30$ in duct flows and simple external flows [6]. If the Reynolds number of the flow is high enough, the distributions in the viscous layer approach the idealizations of constant shear stress and constant heat flux across the layer; therefore, results from a circular tube become applicable to other internal and external flows where this idealization is reasonably valid.

For general insight into and status of the prediction of the turbulent Prandtl number (Pr_t) and its hypothesized functional behavior, the reader is referred to the reviews by Quarmby and Quirk [7], Reynolds [1], Kays [5] and the texts mentioned above or others. A phenomenological explanation of the simplest version is presented by Kays and Crawford [3] and by others; it leads to Reynolds analogy of equal diffusivities for heat and momentum transfer, i.e. $Pr_t = 1$. Scriven [8] pointed out that this result implies impulsive, rather than diffusive, transport of both momentum and energy. (Here we use the term impulsive in the sense of acting momentarily or as in the motion produced by a sudden or momentary force [9] so the energy or momentum is convected rapidly from one region to another, as opposed to 'slower' diffusion.) The idea is that this advection/convection of the fluid from one region to another occurs without time for significant diffusion to/from the parcel/'eddy'. In view of the flow visualization of Corino and Brodkey [10], the impulsive transport corresponds to their 'ejections' rather than the interactions or quiescent phase of readjustment between the sweep and bursting events.

For situations where molecular transport of energy or momentum is expected to be high relative to turbulent transport, a number of investigators have modified the phenomenological derivation to allow for diffusion from the hypothesized eddy during flight and—thereby—have evolved predicted dependencies on the molecular Prandtl number and/or the Reynolds number or related quantities (e.g. Jenkins, [11]). Cebeci [12] extended the reasoning of van Driest [13] to derive a comparable turbulent thermal conductivity and, hence, Pr_t. Although Pr_t is now being predicted via some advanced turbulence models and by direct numerical simulation (DNS) of turbulent flows [14], the current favorites for numerical predictions of practical flows are probably those of Cebeci and of Kays [3].

Unfortunately, it is difficult to measure Pr_t accurately in the region near the wall—where it is important to get it right. As the wall is approached, the experimental uncertainties grow to the point where

✠ Unfortunately, "Bud" Taylor passed away in May 1994.

NOMENCLATURE

b	property variation exponent
c_p	specific heat at constant pressure
D	tube inside diameter
g_c	units conversion factor
h	convective heat transfer coefficient
k	thermal conductivity
q''_w	wall heat flux
T	absolute temperature
V_b	bulk velocity
x	axial location, measured upward from start of heating
y	coordinate perpendicular to the wall.

Non-dimensional quantities

Nu	Nusselt number, e.g. hD/k
Pe_t	turbulent Peclet number, $(\varepsilon/\nu)\,Pr$
Pr	Prandtl number, $c_p\mu/k$; Pr_t, turbulent Prandtl number
Re	Reynolds number, V_bD/ν
y^+	wall distance, $y(g_c\tau_w/\rho)^{1/2}/\nu$.

Greek symbols

ε	eddy diffusivity
μ	absolute viscosity
ν	kinematic viscosity
ρ	density
τ	shear stress; τ_w, wall shear stress.

Subscripts

b	bulk, mixed mean
c	centerline, centerplane
cp	constant property idealization
fd	fully developed (velocity) or fully established (heat transfer)
in	inlet
w	wall.

the measurements cannot be used with confidence to discriminate between hypothesized models [15]. In particular, both the turbulent shear stress and the turbulent heat flux go to zero as the wall is approached, so Pr_t becomes $\sim 0/\sim 0$ and its limiting value is bound to be uncertain, as noted by Bradshaw [16]. Ultimately, at the smallest values of y the turbulent heat flux becomes small relative to molecular transport and Pr_t becomes unimportant.

While direct numerical simulations now give the opportunity for predictions of Pr_t in the viscous layer as shown in Fig. 1 [14, 17–19], they are typically constrained to *low Reynolds number calculations* where the idealizations of constant shear stress and heat flux layers are not valid. For fully developed flow in a tube or parallel plate duct, $\tau\{y\}/\tau_w$ varies as $1-(y^+/y_c^+)$. Therefore, for the predictions of Antonia and Kim [14] and of Kasagi *et al.* [18] at $y^+ \approx 30$, the total shear stress would be reduced to $\tau\{y\}/\tau_w \approx 0.83$ and 0.80, respectively (see Fig. 1 for y^+c). For turbulent boundary layers as treated by Bell and Ferziger, Spalart [20] in an Appendix provides a useful discussion of $d\tau\{y\}/dy$ for the inner layer; Bradshaw [16] notes that in predicting the mean velocity profile, the effect of the $\tau\{y\}$ variation is always less than for the equivalent pipe flow. Further, DNS are so expensive that only a few values of molecular Prandtl number have been treated.

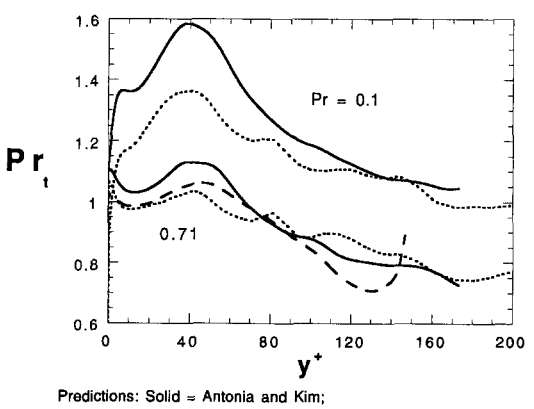

Fig. 1. Distribution of the turbulent Prandtl number as predicted by recent direct numerical simulations. Solid lines = Antonia and Kim [14], dashed line = Kasagi *et al.* [18] and dotted lines = Bell and Ferziger [19].

A few years ago, the authors suggested that one can take advantage of the sensitivity of the local Nusselt number for the thermal entry in tube flow to deduce an effective turbulent Prandtl number for the viscous layer [21]. The idea was that while the thermal boundary layer is growing across the viscous layer the heat transfer coefficient must be controlled by the value of Pr_t in that region; then further downstream Nu could be affected by Pr_t at larger distances from the wall. McEligot *et al.* [21] demonstrated the degree of sensitivity of $Nu\{x\}$ to various hypothesized forms of $Pr_t\{y\}$. Conceptually, one could deduce $Pr_t\{y\}$ from the measured $Nu\{x\}$ distribution. Alternatively, data from a carefully controlled experiment with a smooth vertical circular tube can be used to test the adequacy of hypothesized models of $Pr_t\{y^+, Pr, Re\}$, the main thrust of the present study.

The *purpose* of the present work is to examine the validity, for 'high' Reynolds numbers, of the dependence on molecular Prandtl number as implied by the new predictions of Pr_t from the direct numerical simulations and of some of the popular Pr_t models. Our route is via comparison to the authors's measurements of $Nu_{cp}\{x/D\}$ for $0.18 < Pr < 0.72$ with gas mixtures [22]. In a mathematical sense this is a necessary, but not sufficient, test of the behavior of Pr_t;

in practice it may essentially be a sufficient test for predictions of the wall temperatures and wall heat fluxes usually desired by the thermal design engineer.

There are fairly extensive heat transfer data reported for liquid metals, which have low Prandtl numbers. In the present paper, we avoid their consideration for a number of reasons. Liquid metal measurements are generally thought to be unreliable; this observation is partially an outcome of the low temperature differences involved in a 'constant properties' experiment so that small errors in temperature measurement lead to large errors in Nusselt numbers. Consequently, there is the danger that systematic experimental problems might be interpreted as variations of Pr_t. Due to the high thermal conductivity of liquid metals, their Nusselt numbers are not very sensitive to k_{turb} (therefore Pr_t); thus, greater precision in measuring Nu is needed to deduce the equivalent Pr_t within a given uncertainty than with gases. And it is the perception of one of the authors (DMM) that in liquid metal flows, the convective thermal resistance is so low that other thermal resistances tend to dominate in practical applications.

2. EXPERIMENTS

A range of molecular Prandtl numbers may be attained by mixing a heavy gas such as argon or xenon with a light one like hydrogen or helium [23]. Therefore, the molecular Prandtl number can be varied to optimize performance or weight of closed cycle gas turbine systems. Knowledge of such gas mixtures is also important for gas pipelines since Pr is typically about 1/3 for some of the gas mixtures involved.

Data were obtained with mixtures of helium and argon, helium and xenon, hydrogen and carbon dioxide, hydrogen and xenon and of nitrogen and oxygen (called air) to cover ranges $0.18 < Pr < 0.72$ and $3 \times 10^4 < Re_{in} < 1 \times 10^5$. Tabulated results are provided in the reports of Pickett [24], Serksnis [25] and the present authors [26]. For the present paper, measurements at 'high' Reynolds number were concentrated from $Re_{in} \approx 3.4 \times 10^4$ to $Re_{in} \approx 8.4 \times 10^4$.

Details of the apparatus and procedures are presented in the papers by Pickett and Serksnis and their reports cited therein plus the paper by the present authors [22, 27, 28]. Essentially, the configuration gave internal turbulent flow of the gas mixture through a small vertical tube heated resistively. An unheated entry region preceded the electrode to allow a fully-developed velocity profile to be approached, thereby setting up the classical thermal entry problem. For a chosen mixture and at a given entry Reynolds number, data were obtained at a series of heating rates and local heat transfer parameters (such as $Nu\{x/D, T_w/T_b\}/Re^{0.8}Pr^{0.4}$) were deduced; the properties were evaluated at the local bulk temperature at each point. These values were then extrapolated to $T_w/T_b = 1$ in order to deduce the local values corresponding to the constant properties idealization, e.g. $Nu_{cp}\{x/D, Re_{in}, Pr\}$. Although the maximum temperature ratio was about 2.4, the extrapolation was typically only from $T_w/T_b \approx 1.1$ to unity as shown in Fig. 4 of the paper by Taylor *et al.* [22]. From adiabatic pressure drop measurements, the experimental friction factors were found to agree with the accepted correlation of Drew *et al.* [29] within 2% (we use this correlation because the friction factor is an explicit function of the Reynolds number and it does not differ significantly from the oft-cited, implicit Karman–Nikuradse relation [3, equation 11-11]).

The tube diameter of 5.87 mm (0.231 in) was selected to reduce the ratio between the heat losses and the heat transfer to the flowing gas and to reduce Gr/Re^2 to approach 'pure' forced convection, the usual idealization. Unfortunately, this small size precludes introducing probes into the flow to infer $Pr_t\{y\}$; however, as demonstrated by Simpson *et al.* [15], the experimental uncertainties in deducing $Pr_t\{y\}$ from profile measurements typically increase extremely as the wall is approached and, thereby, reduce the validity of the results.

Experimental uncertainties were estimated by the technique of Kline and McClintock [30] followed by graphical extrapolation to $T_w/T_b = 1$ at a number of axial locations. Typical uncertainties for the Nusselt number were about 8% at $x/D \approx 1.3$ decreasing to about 5% at $x/D > 24$. These estimates of experimental uncertainties do not include the contributions of the uncertainties in the fluid properties themselves. Thus, for the comparisons it is implied that the evaluation of the mixture properties is included in the numerical models in the same manner as derived by the present authors (e.g. see ref. [22]).

3. PREDICTIVE TECHNIQUES

For gas mixtures with Pr ranging from about 0.18 to 0.7, predicted heat transfer parameters are expected to be strongly dependent on the representation of thermal energy transport in the viscous layer, $y^+ < \sim 30$ [21]. This expectation evolves since, in a typical high Reynolds number flow, about 40% of the thermal resistance can be concentrated in the region $5 < y^+ < 30$, which covers only about 0.2% of the radius or boundary layer thickness. Therefore, it is important that any turbulence model (or large eddy simulation) provides predictions representing this behavior reasonably and that direct numerical simulations be checked for application to realistic flows. (For $y^+ < 5$ in a gas, molecular momentum and energy transport dominate over their turbulent counterparts so determination of Pr_t becomes unimportant—hence the earlier description of this region as a 'laminar' sublayer.)

To examine the validity of models for $Pr_t\{y^+, Pr\}$, we obtained predictions by applying the computer program of Bankston and McEligot [31] to the conditions of the experiments. This program is a finite control volume technique for internal turbulent flow

with fluid property variation, usually employing a mixing length model for predicting effective viscosities. The grid spacing expands from the wall with the first node falling at $y^+ \approx 0.5$ or less. For the present calculations, property variation was suppressed by setting exponents in power law approximations (e.g. $k \sim T^b$) to zero. The effective viscosity is predicted herein with a combination of the popular van Driest [13] mixing length and the Reichardt middle law [32]. With $\kappa = 0.4$ and $A^+ = 26$, this representation yields adiabatic friction factors within about 1% of the empirical correlation of Drew *et al.* [29]. Node spacing was adjusted to provide convergence of Nusselt number predictions to within about 1/2%.

For the thermal entry comparisons shown herein, the deduced wall heat flux variation $q''_w\{x\}$ of the experiments was employed as the boundary condition, rather than the usual constant-heat-flux idealization. Electrical resistance heating of the test section, as in this experiment, yields an axial heat flux distribution near the electrode that may be approximated as exponentially approaching a constant value at small axial distances. For the data of Taylor *et al.* [22] in Fig. 3, after 1–1/2 diameters q''_w was constant to within 2% until near the downstream electrode. Reynolds *et al.* [33] provide an eigenvalue solution for the exponential description; for their low-Reynolds-number data this analytical solution approaches the comparable solution based on the constant-wall-heat-flux idealization to within 2% before four diameters.

Unfortunately, predictions of $Pr_t\{y^+, Pr\}$ from direct numerical simulations are not available for Pr between 0.18 and 0.7, the range of data with the gas mixtures. And apparently direct numerical simulations are too expensive to repeat the calculations at the Prandtl number of the existing data. Therefore, we derived an estimate by approximate interpolation from the predictions at $Pr = 0.1$ and 0.71 by Antonia and Kim [14] and $Pr = 0.025$ and 0.71 by Kasagi and Ohtsubo [17] and Kasagi *et al.* [18] as follows. If one examines Fig. 1, one sees common features in each prediction of $Pr_t\{y^+\}$: typically they vary sharply with y^+ to $y^+ \approx 5$ (a region where Pr_t is not important for gases because molecular transport dominates), vary more gradually to a peak at $y^+ \approx 40$, then drop again to $y^+ \approx 150$ after which they are approximately constant. We approximated each of these curves as linear between these characteristic locations; over the range $Pr = 0.025$–0.71, the values of the vertices, such as $Pr_t\{y^+ = 5, Pr\}$, were found to vary in a loosely logarithmic manner. These observations were used to provide piecewise linear estimates of $Pr_t\{y^+, Pr\}$ from power law representations of the values of the vertices. These representations were *approximate* graphical fits for Pr_t at each vertex, determined from the predictions at $Pr = 0.025$ [17, 18] and 0.1 [4] and the two predictions at $Pr = 0.71$ (both). A typical example is shown for $Pr = 0.18$ in Fig. 2.

Other models tested were those of Cebeci and of Kays plus constant values of $Pr_t\{y^+\}$, such as Reynolds analogy. The Cebeci model was taken from equations (6.25) and (6.26) in the text by Cebeci and Bradshaw [2] with the constants taken as $C_4 = 6.33$ and $\kappa_h = 0.44$ from his original paper [12] and its discussion by Na and Habib. The Kays model was evaluated via equation (13-7) in the 3rd edition of the text by Kays and Crawford [4] with $C = 0.3$ and $Pr_{t,\infty} = 0.85$, their revised constants.

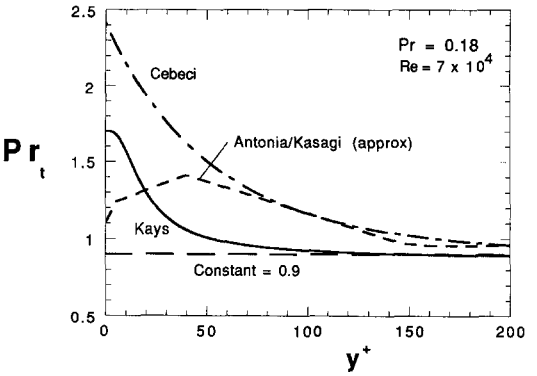

Fig. 2. Predictions of the turbulent Prandtl number from various models evaluated for a mixture of hydrogen and xenon ($Pr \approx 0.18$).

Figure 2 compares four models for $Pr_t\{y^+, Pr\}$ at $Re = 7 \times 10^4$ and $Pr = 0.18$, corresponding to a mixture of hydrogen and xenon. There are significant differences at low y^+ and all four are in approximate agreement at large y^+. Above $y^+ \approx 100$ the Kays model is close to the constant model. The approximation derived from the direct numerical simulations of Antonia and Kim and of Kasagi and coworkers agree with that of Cebeci at high y^+, then they diverge at lower values. For these conditions the constant shear layer approximation becomes invalid by more than 5% for $y^+ > \sim 80$; therefore, the 'log law' idealization could be reasonable for the range $30 < y^+ < 80$. As suggested by Kays [5], molecular transport would dominate for $(\varepsilon_m/\nu)Pr/Pr_t < 0.05$, which would occur at $y^+ < \sim 7$ with Reynolds analogy applied. The resulting predictions for the wall heat transfer parameters are presented in Fig. 3a.

4. RESULTS

4.1. *Thermal development*

Comparisons of the heat transfer measurements and the predictions of the Pr_t models are shown for three gas mixtures in Fig. 3. The Dittus–Boelter [34] correlation is employed primarily as a convenient nondimensionalization, rather than as a standard of comparison. However, it and the Colburn [35] analogy are still in use in industry. For gases at high Reynolds number, the more recent correlation of Gnielinski [36] is effectively the same as that of Dittus and Boelter.

All the data in Fig. 3 demonstrate that the Dittus–Boelter equation for gases at large x/D [37] is inadequate for these low Prandtl numbers. (If the Dittus–

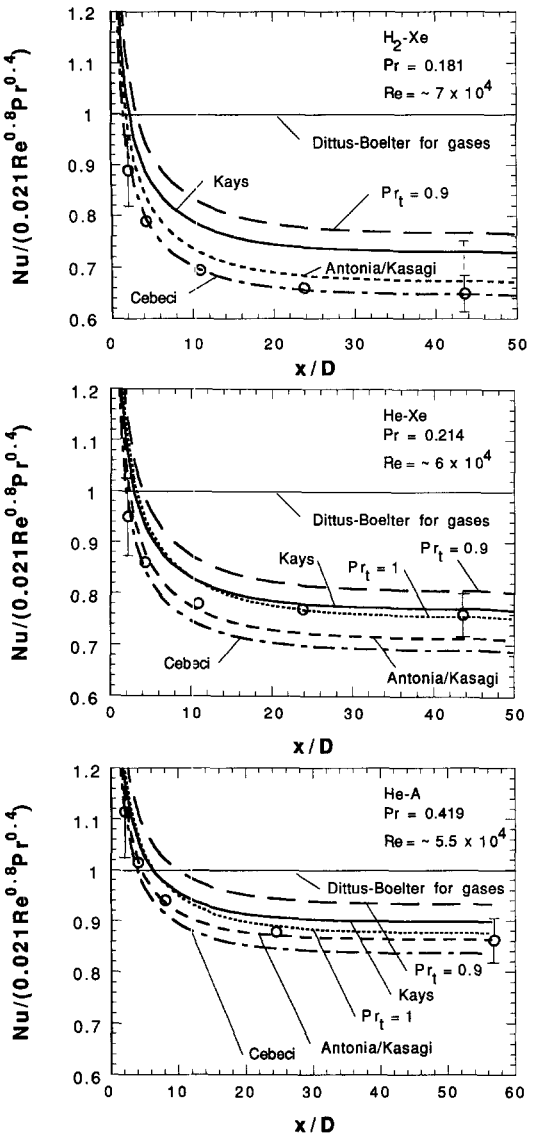

Fig. 3. Comparison of measurements of wall heat transfer parameters for internal flow of gas mixtures to numerical predictions based on various models for the turbulent Prandtl number.

Boelter equation for liquids is used, the coefficient of 0.023 gives a 10% worse discrepancy.) This conclusion was also presented in our earlier paper [22].

As noted by McEligot *et al.* [21], numerical results in the idealized thermal entry correspond to the Leveque solution for constant wall heat flux [38] until the thermal boundary layer grows beyond the 'linear' sublayer. Then (after $(\varepsilon_m/\nu)Pr/Pr_t > \sim 0.05$) the differences in the models for Pr_t become significant in the near wall region. As the flow proceeds downstream, the turbulence properties at successively greater distances from the wall become important progressively.

Figure 3a, b and c provides local comparisons for mixtures of hydrogen with xenon, helium with xenon and helium with argon, respectively. As the molecular Prandtl number increases, the data and predictions downstream approach the Dittus–Boelter correlation for gases, which is reasonable for Pr near the 0.7 value of common gases. Also the difference between the highest and lowest predictions decreases slightly.

For $Pr = 0.18$ the four models only differ significantly to $y^+ \approx 150$ (Fig. 2); however, these differences still affect the downstream behavior. At $x/D \approx 40$, the $Nu\{x\}$ predictions from the highest model ($Pr_t = 0.9$) are about 12% higher than those from the lowest (Cebeci). Since experimental uncertainties are normally estimated to be only about 5% there, one could conceptually discriminate between the two models. However, the comparisons from the one set of measurements for a hydrogen–xenon mixture ($Pr \approx 0.18$) are shown only for completeness. As indicated by Taylor *et al.* [22], the transport properties of this mixture are more uncertain than those of helium–xenon mixtures. In particular, an alternate method of estimating the thermal conductivity [39] would lead to measured Nusselt numbers 6–10% higher than shown; the dashed uncertainty bracket at $x/D \approx 44$ has been added to account for this possibility. Depending on which properties are employed, the data agree best with predictions from the Cebeci model or with those from the Kays model; accurate measurement of the transport properties of the actual mixture would be needed to resolve this choice.

For the helium–xenon mixture at $Pr \approx 0.21$, agreement with the measurements is good for Reynolds analogy and the Kays model which are almost coincident with each other. From Fig. 2 (at a slightly lower Pr), one sees that the values of Pr_t from the Kays model are higher than Reynolds analogy for $y^+ < \sim 50$ and lower for $y^+ > 50$. In the consequent predictions shown in Fig. 3b, one can see that at low x/D this effect leads to slightly lower Nusselt numbers predicted from the Kays model and that the roles are reversed further downstream. At this Prandtl number the predictions for $Pr_t = 0.9$ and the DNS approximation also are almost within the bounds of the estimated experimental uncertainty. (An experimental uncertainty of about 6% in the Nusselt number corresponds to an uncertainty in an effective Pr_t of about 10%.)

It might be considered that there is an inconsistency between the predictions at $Pr \approx 0.18$ and 0.21, and Fig. 3a, b. Since the difference in Pr is apparently small, one expects that Nu would be higher for $Re \approx 7 \times 10^4$ than 6×10^4. However, normalization by $Re^{0.8}$ accounts for most of the effect. In the next section it is shown that there is predicted to be about a 10% decrease in Nu_{fd} from $Pr = 0.214$ to 0.181 at $Re \approx 6 \times 10^4$. The change of $Re^{0.8}$ from $Re \approx 7 \times 10^4$ to 6×10^4 is about 13%, so the effects approximately counter one another.

For the helium–argon mixture ($Pr \approx 0.42$), the data of Pickett [24] agree best with the approximation of the DNS results and with Reynolds analogy ($Pr_t = 1$), but the estimated experimental uncertainty en-

compasses predictions from all models from that of Kays to that of Cebeci. Only the model using a constant Pr_t of 0.9 differs significantly.

From the level of agreement in Fig. 3, one might conclude that the Kays model of Fig. 2 is too low in the near-wall region and is reasonable further out. That is, for low axial distances while the thermal boundary layer is growing through the near-wall region, the Pr_t models of Cebeci and of Antonia/Kasagi appear to give better agreement with the measured local Nusselt numbers. Further downstream, as the thermal boundary layer extends to the centerline, the Kays model gives reasonable agreement with the data. Although ideally one could use these thermal entry data in this manner to deduce $Pr_t\{y^+\}$, in the experiments axial conduction in the tube wall and other problems interfere. Further, the predictions of the Kays model are generally within the experimental uncertainty band even at low x/D, so the present authors doubt that such discrimination would be warranted.

Kays [40] recently compared our data from Fig. 3 to his newer correlations for Pr_t using a constant wall heat flux approximation. He found close agreement between his liquid metal version, $Pr_t = (2.0/Pe_t)+0.85$, and our measurements for $Pr \approx 0.18$ but less so for the other two runs. For $Pr \approx 0.21$ and 0.42 his relationship from fitting DNS calculations, $Pr_t = (0.7/Pe_t)+0.85$, matches our data better than the liquid metal version, particularly downstream. As a consequence of the level of agreement found in the thermal entry, he concluded that in all cases it does appear that $Pr_t\{Pe_t\}$ provides results that are equally good when the thermal boundary layer is mostly in the sublayer or when it is fully developed.

4.2. *Fully-established conditions*

In order to obtain more complete insight, the data for the almost fully-established conditions presented by Taylor *et al.* [22] in their Fig. 5 have also been applied for comparisons. These data have been adjusted (slightly) to three common values for Reynolds number, as described in their paper. These deduced data and their estimated uncertainties are listed in Table 1 for the convenience of readers who may want to make their own comparisons.

For each value of Reynolds number, predictions of Nu_{cp} were generated for the range, $0.1 \leqslant Pr \leqslant 1$, with each Pr_t model. As seen in Fig. 3, the values at $x/D = 55$ correspond to fully-established heat transfer as defined by Kays and Crawford [3]. These predictions are presented in Fig. 4 with the deduced data of Taylor *et al.* (i.e. Table 1).

As implied earlier in Section 4.1, the popular Dittus–Boelter correlation with the coefficient of 0.21 for gases only agrees with the data for the Prandtl number range of common gases; likewise, it is in reasonable agreement with the predictions of the various Pr_t models only in this range. Otherwise, the exponent on the Prandtl number is not large enough to predict the proper magnitudes as it decreases. Consequently, the correlation overpredicts the Nusselt number and, therefore, predicted convective thermal resistances would be too low. These same comments can be made concerning the Colburn analogy. This situation is dangerous in many applications.

Table 1. Fully-established, constant-property Nusselt numbers for low-Prandtl-number gas mixtures as deduced from the data of Taylor *et al.* [22, 26]

Prandtl number	Nusselt number	Estimated experimental uncertainty
	$Re = 3.4 \times 10^4$	
0.214	37.0	2.2
0.251	42.9	2.7
0.301	45.2	2.2
0.34	51.9	2.5
0.419	55.8	2.3
0.486	61.9	2.7
0.667	75.3	3.7
0.717	74.2	4.0
	$Re = 6.0 \times 10^4$	
0.214	57.3	3.2
0.231	61.1	3.6
0.419	84.0	3.2
0.717	114.	5.8
	$Re = 8.4 \times 10^4$	
0.181	61.0	4.9
0.251	76.1	4.3
0.34	98.9	4.7
0.486	123.	5.8
0.717	151.	7.2

With these same data Taylor *et al.* [22] demonstrated good agreement with Nu_{fd} correlations by Petukhov [41] and by Kays [42] for low Prandtl number gas mixtures. Relations by Churchill [43] and Sleicher and Rouse [44] did not agree as well at low Prandtl numbers.

The predictions based on the Kays Pr_t model are quite close to those from Reynolds analogy; they agree with one another within about 3% over this range of comparisons. For the three Reynolds numbers, at low Prandtl numbers the sequence of the predictions is the same except for the Kays model relative to Reynolds analogy, i.e. Cebeci < DNS < Kays < Pr_t of 0.9, respectively. (Near the common gases, the values representing DNS converge with and then cross the predictions for Kays.) If one examines Fig. 2, one sees that the general sequence of predictions at low Prandtl number corresponds to the levels of the Pr_t models in the range $20 < y^+ < 60$, the range where turbulent momentum transport begins to be dominant compared to molecular transport.

The models may be evaluated by comparing their predictions directly to the deduced data, taking into account the estimated experimental uncertainties of the data (identified by the vertical brackets). For $Re = 6.0 \times 10^4$ and 8.4×10^4, results for $Pr_t = 0.9$ are usually slightly too high relative to the bounds of the uncertainties. For some experimental conditions, the

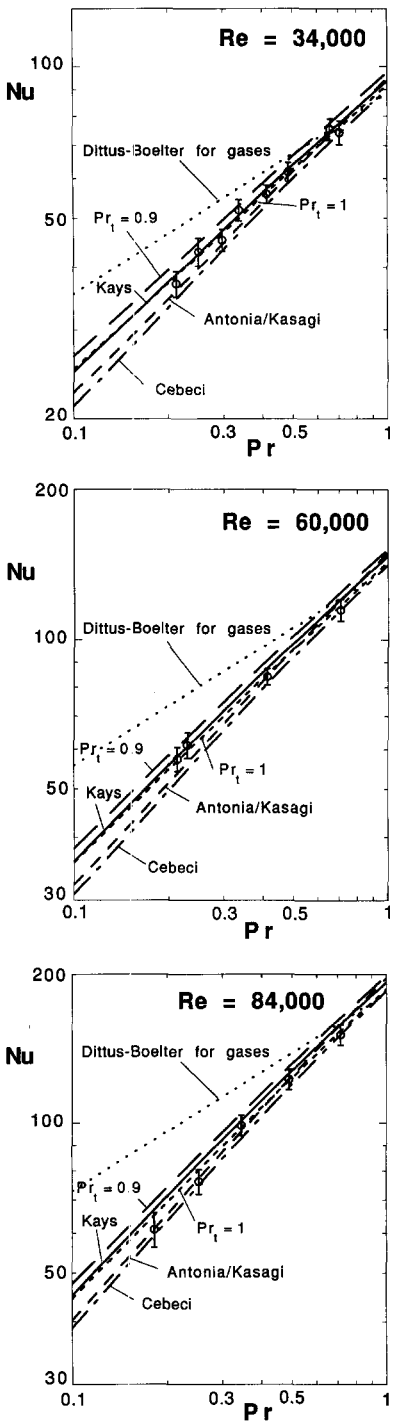

Fig. 4. Comparisons of deduced Nusselt number measurements for constant properties and fully-established thermal conditions to numerical predictions.

version from Cebeci is slightly outside the uncertainty brackets. On average, the deduced Nusselt numbers (i.e. the points themselves) favor predictions from Reynolds analogy and from Kays, but our approximation of the DNS models of Antonia and Kasagi with their colleagues (labeled 'Antonia/Kasagi') fall within the range of experimental uncertainties in all cases. Only the less certain data with hydrogen ($Pr \approx 0.18$ and 0.33) at $Re = 8.4 \times 10^4$ tend to agree with the Cebeci model better. (For $Pr > 0.2$, the differences in Nu_{fd} between Antonia/Kasagi and Kays are less than about 8%.)

At $Re = 3.4 \times 10^4$, the experimental scatter is slightly greater than at the higher Reynolds numbers, but it is not greater than the envelope of the estimated experimental uncertainties. For all data points, the versions for Kays and Reynolds' analogy fall within the estimated uncertainties and the Antonia/Kasagi prediction is clearly outside only for one. The two extreme models are outside the uncertainty ranges of several points—Cebeci barely and $Pr_t = 0.9$ more so—but not for all.

Overall, there are not large differences between predictions. At $Pr \approx 0.7$ for common gases, the authors of the models had the benefit of extensive data when developing their empirical constants/functions. At low Prandtl number, the sensitivity to choice of model is not great because the molecular thermal conductivity becomes more comparable to the turbulent thermal conductivity in the viscous layer ($y^+ < \sim 30$), where the dominant thermal resistance tends to be located. This latter observation is consistent with Kays' [5] identification of the grouping $[(1/Pr) + (\varepsilon_m/Pr_t\, \nu)]$ or $(Pr\varepsilon_m/(\nu Pr_t))$ as an indicator of the relative importance, or lack thereof, of turbulent energy transport.

4.3. *Summary*

On balance, the model of Kays or the Reynolds analogy appeared to fare best overall for the molecular Prandtl number range and Reynolds numbers of our comparisons. These results are consistent with the observations of Pickett and of Serksnis who found an 'effective Pr_t' of unity to be adequate for $Pr \approx 0.49$ and 1/3, respectively. As indicated earlier, this observation implies that—even at these low Prandtl numbers—the turbulent transport that matters can be considered to be effectively impulsive (or 'convective') without significant thermal conduction from the turbulent structures observed during the bursting process by Corino and Brodkey [10] and Blackwelder and Kaplan [45]. However, the predictions from the approximations of the low Reynolds number DNS fared reasonably well; it is anticipated that a comparable approximation based on the DNS of Bell and Ferziger [19] at higher momentum-thickness Reynolds numbers would agree even better. On the other hand, the present authors would not seriously contest a reviewer's claim that the data favor no model over the others (except over $Pr_t = 0.9$). All models examined predicted the *trends* of the measurements with respect to molecular Prandtl number.

5. CONCLUDING REMARKS

In order to examine the validity of hypothesized distributions of the turbulent Prandtl number, $Pr_t\{y^+, Pr, Re\}$, numerical predictions for the tur-

bulent thermal entry problem and downstream have been compared to measurements of wall heat transfer parameters for flow of low-Prandtl-number gas mixtures in small tubes. Models of Pr_t employed included approximations of recent results from direct numerical simulations of turbulent heat transfer, as well as those of Kays and of Cebeci, a constant effective value of 0.9 and Reynolds analogy. Data for a range of molecular Prandtl numbers for 'high' Reynolds numbers, $\sim 3 \times 10^4 < Re_{in} < \sim 1 \times 10^5$, were obtained earlier by Taylor *et al.* [22] by utilizing mixtures formed by helium or hydrogen with xenon, argon or carbon dioxide. All models examined predicted the trends of these data with respect to molecular Prandtl number. For the range $0.21 < Pr < 0.72$, Reynolds analogy and the variable $Pr_t\{y^+, Pr, Re\}$ model by Kays agreed with downstream measurements to within about 5%, which should be adequate for most thermal designs.

The apparatus was designed specifically to use gas as the working fluid and this choice limited the molecular Prandtl number to a range from 0.2 to 0.7, approximately. As noted by a reviewer, 'a major conclusion is that the turbulent Prandtl number in the range $10 < y^+ < 60$ most influences the predictions. The region close to the wall is unimportant for the range of Prandtl numbers considered. More light could be shed on the near-wall region by looking at fluids with $Pr > 1$. Similarly, the region $y^+ > 60$ would come into play if data with $Pr < 0.1$ were considered.' We agree. While beyond the scope of the present study (and beyond the precision of the present apparatus and technique if used for liquids), further efforts for $Pr > 1$ would be warranted, particularly for thermal entry problems in oils. To the authors' knowledge, the highest value available with direct numerical simulation is $Pr = 2$. As pointed out by Kays [5], the grouping $(Pr\varepsilon_m/(\nu Pr_t))$ is an indicator of the relative importance, or lack thereof, of turbulent energy transport—so knowledge of the variation of Pr_t becomes more important as Pr is increased and, conversely, less for liquid metals.

Acknowledgements—The study reported was partly supported through the Laboratory Directed Research and Development Program and the Long Term Research Initiative Program of EG&G Idaho and Lockheed Idaho Technologies Company under DoE Idaho Field Office contracts DE-AC07-76ID01570 and DE-AC07-94ID13223, respectively. Earlier related experiments and their analysis were financed by the Office of Naval Research (Mr M. Keith Ellingsworth, Program Manager), Gould Ocean Systems Division and the Aerospace and Mechanical Engineering Department of the University of Arizona. Professors R. A. Antonia of the University of Newcastle, J. H. Ferziger of Stanford University and N. Kasagi of Tokyo University very kindly provided tabulated results from their direct numerical simulations. Professors W. M. Kays and P. Bradshaw of Stanford and three journal referees made useful comments on the initial draft of the paper. To all we are extremely grateful. By acceptance of this article for publication, the publisher recognizes the U.S. Government's (license) rights in any copyright and the Government and its authorized representatives have unrestricted right to reproduce in whole or in part said article under any copyright secured by the publisher.

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